

# Stochastic processes: basic notions \*

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## 1. Fundamental concepts

### 1.1. Probability space

**Definition 1.1** PROBABILITY SPACE. A probability space is a triplet  $(\Omega, \mathcal{A}, \mathbb{P})$  where

- (1)  $\Omega$  is the set of all possible results of an experiment;
- (2)  $\mathcal{A}$  is a class of subsets of  $\Omega$  (called events) forming a  $\sigma$ -algebra, i.e.

$$(i) \Omega \in \mathcal{A} ,$$

$$(ii) A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} ,$$

$$(iii) \bigcup_{j=1}^{\infty} A_j \in \mathcal{A} , \text{ for any sequence } \{A_1, A_2, \dots\} \subseteq \mathcal{A} ;$$

- (3)  $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$  is a function which assigns to each event  $A \in \mathcal{A}$  a number  $\mathbb{P}(A) \in [0, 1]$ , called the probability of  $A$  and such that

$$(i) \mathbb{P}(\Omega) = 1 ,$$

$$(ii) \text{ if } \{A_j\}_{j=1}^{\infty} \text{ is a sequence of disjoint events, then } \mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j) .$$

## 1.2. Real random variable

**Definition 1.2** REAL RANDOM VARIABLE (HEURISTIC DEFINITION). A real random variable  $X$  is a variable with real values whose behavior can be described by a probability distribution. Usually, this probability distribution is described by a distribution function:

$$F_X(x) = \mathbb{P}[X \leq x]. \quad (1.1)$$

**Definition 1.3** REAL RANDOM VARIABLE. A real random variable  $X$  is a function  $X : \Omega \rightarrow \mathbb{R}$  such that

$$X^{-1}((-\infty, x]) := \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{A}, \forall x \in \mathbb{R}.$$

$X$  is a measurable function. The probability distribution of  $X$  is defined by

$$F_X(x) = \mathbb{P}[X^{-1}((-\infty, x])]. \quad (1.2)$$



### 1.3. Stochastic process

**Definition 1.4** REAL STOCHASTIC PROCESS. *Let  $\mathbb{T}$  be a non-empty set. A stochastic process on  $\mathbb{T}$  is a collection of random variables  $X_t : \Omega \rightarrow \mathbb{R}$  such that a random variable  $X_t$  is associated with each element  $t \in \mathbb{T}$ . This stochastic process is denoted by  $\{X_t : t \in \mathbb{T}\}$ , or more simply by  $X_t$  when the definition of  $\mathbb{T}$  is clear. If  $\mathbb{T} = \mathbb{R}$  (real numbers),  $\{X_t : t \in \mathbb{T}\}$  is a continuous time process. If  $\mathbb{T} = \mathbb{Z}$  (integers) or  $\mathbb{T} \subseteq \mathbb{Z}$ ,  $X_t : t \in \mathbb{T}\}$  is discrete time process.*

The set  $\mathbb{T}$  can be finite or infinite, though usually it is more convenient to assume that  $\mathbb{T}$  is infinite. In the sequel, we focus on processes where  $\mathbb{T}$  is a right-infinite interval of integers, i.e.

$$\mathbb{T} = (n_0, \infty) \text{ where } n_0 \in \mathbb{Z} \text{ or } n_0 = -\infty. \quad (1.3)$$

We can also consider random variables which take their values in more general spaces, i.e.

$$X_t : \Omega \rightarrow \Omega_0 \quad (1.4)$$

where  $\Omega_0$  is any non-empty set. Unless stated otherwise, we shall limit ourselves to the case where  $\Omega_0 = \mathbb{R}$ . To observe a time series is equivalent to observing a realization of a process  $\{X_t : t \in \mathbb{T}\}$  or a portion of such a realization: given  $(\Omega, \mathcal{A}, \mathbb{P})$ ,  $\omega \in \Omega$  is drawn first, and then the variables  $X_t(\omega)$ ,  $t \in \mathbb{T}$ , are associated with it. Each realization is determined in one shot by  $\omega$ .

The probability law of a stochastic process  $\{X_t : t \in \mathbb{T}\}$  with  $\mathbb{T} \subseteq \mathbb{R}$  can be described by specifying the joint distribution function of  $(X_{t_1}, \dots, X_{t_n})$  for each subset  $\{t_1, t_2, \dots, t_n\} \subseteq \mathbb{T}$  (where  $n \geq 1$ ):

$$F(x_1, \dots, x_n; t_1, \dots, t_n) = \mathbb{P}[X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n]. \quad (1.5)$$

This follows from Kolmogorov's theorem [see Brockwell and Davis (1991, Chapter 1)].

#### 1.4. $L_r$ spaces

**Definition 1.5**  $L_r$  SPACE. *Let  $r$  be a real number.  $L_r$  is the set of real random variables  $X$  defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  such that  $\mathbb{E}[|X|^r] < \infty$ .*

The space  $L_r$  is always defined with respect to a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . In other words, we have:

$$L_r = L_r(\Omega, \mathcal{A}, \mathbb{P}). \quad (1.6)$$

$L_2$  is the set of random variables on  $(\Omega, \mathcal{A}, \mathbb{P})$  whose second moments are finite (*square-integrable variables*). A stochastic process  $\{X_t : t \in \mathbb{T}\}$  is in  $L_r$  iff  $X_t \in L_r, \forall t \in \mathbb{T}$ , i.e.

$$\mathbb{E}[|X_t|^r] < \infty, \forall t \in \mathbb{T}. \quad (1.7)$$

The properties of moments of random variables are summarized in Dufour (2022).

## 2. Stationary processes

In general, the variables of a process  $\{X_t : t \in \mathbb{T}\}$  are not identically distributed nor independent. In particular, if we suppose that  $\mathbb{E}(X_t^2) < \infty$ , we have:

$$\mathbb{E}(X_t) = \mu_t, \tag{2.1}$$

$$\text{Cov}(X_{t_1}, X_{t_2}) = \mathbb{E}[(X_{t_1} - \mu_{t_1})(X_{t_2} - \mu_{t_2})] = C(t_1, t_2). \tag{2.2}$$

The means, variances and covariances of the variables of the process depend on their position in the series. The behavior of  $X_t$  can change with time. The function  $C : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$  is called the *covariance function* of the process  $\{X_t : t \in \mathbb{T}\}$ .

In this section, we will focus on the case where  $\mathbb{T}$  is a right-infinite interval of integers.

### Assumption 2.1 PROCESS ON AN INTERVAL OF INTEGERS.

$$\mathbb{T} = \{t \in \mathbb{Z} : t > n_0\} \quad \text{where } n_0 \in \mathbb{Z} \cup \{-\infty\}. \tag{2.3}$$

**Definition 2.1** STRICTLY STATIONARY PROCESS. *A stochastic process  $\{X_t : t \in \mathbb{T}\}$  is strictly stationary (SS) iff the probability distribution of the vector  $(X_{t_1+k}, X_{t_2+k}, \dots, X_{t_n+k})'$  is identical with the one of  $(X_{t_1}, X_{t_2}, \dots, X_{t_n})'$ , for any finite subset  $\{t_1, t_2, \dots, t_n\} \subseteq \mathbb{T}$  and any integer  $k \geq 0$ . To indicate that  $\{X_t : t \in \mathbb{T}\}$  is SS, we write  $\{X_t : t \in \mathbb{T}\} \sim SS$  or  $X_t \sim SS$ .*

**Proposition 2.1** CHARACTERIZATION OF STRICT STATIONARITY FOR A PROCESS ON  $(n_0, \infty)$ . *If the process  $\{X_t : t \in \mathbb{T}\}$  is SS, then the probability distribution of the vector  $(X_{t_1+k}, X_{t_2+k}, \dots, X_{t_n+k})'$  is identical to the one of  $(X_{t_1}, X_{t_2}, \dots, X_{t_n})'$ , for any finite subset  $\{t_1, t_2, \dots, t_n\} \subseteq \mathbb{T}$  and any integer  $k > n_0 - \min\{t_1, \dots, t_n\}$ .*

For processes on the integers ( $\mathbb{T} = \mathbb{Z}$ ), the above characterization can be formulated in a simpler way as follows.

**Proposition 2.2** CHARACTERIZATION OF STRICT STATIONARITY FOR A PROCESS ON THE INTEGERS. *A process  $\{X_t : t \in \mathbb{Z}\}$  is SS iff the probability distribution of  $(X_{t_1+k}, X_{t_2+k}, \dots, X_{t_n+k})'$  is identical with the probability distribution of  $(X_{t_1}, X_{t_2}, \dots, X_{t_n})'$ , for any subset  $\{t_1, t_2, \dots, t_n\} \subseteq \mathbb{Z}$  and any integer  $k$ .*

**Definition 2.2** SECOND-ORDER STATIONARY PROCESS. *A stochastic process  $\{X_t : t \in \mathbb{T}\}$  is second-order stationary (S2) iff*

$$\begin{aligned}
 (1) \quad & \mathbb{E}(X_t^2) < \infty, \forall t \in \mathbb{T}, \\
 (2) \quad & \mathbb{E}(X_s) = \mathbb{E}(X_t), \forall s, t \in \mathbb{T}, \\
 (3) \quad & \text{Cov}(X_s, X_t) = \text{Cov}(X_{s+k}, X_{t+k}), \forall s, t \in \mathbb{T}, \forall k \geq 0.
 \end{aligned}
 \tag{2.4}$$

*If  $\{X_t : t \in \mathbb{T}\}$  is S2, we write  $\{X_t : t \in \mathbb{T}\} \sim S2$  or  $X_t \sim S2$ .*

**Remark 2.1** Instead of second-order stationary, we also say weakly stationary (WS).

**Proposition 2.3** RELATION BETWEEN STRICT AND SECOND-ORDER STATIONARITY. *If the process  $\{X_t : t \in \mathbb{T}\}$  is strictly stationary and  $\mathbb{E}(X_t^2) < \infty$  for any  $t \in \mathbb{T}$ , then the process  $\{X_t : t \in \mathbb{T}\}$  is second-order stationary.*

PROOF. Suppose  $\mathbb{E}(X_t^2) < \infty$ , for any  $t \in \mathbb{T}$ . If the process  $\{X_t : t \in \mathbb{T}\}$  is SS, we have:

$$\mathbb{E}(X_s) = \mathbb{E}(X_t), \forall s, t \in \mathbb{T}, \quad (2.5)$$

$$\mathbb{E}(X_s X_t) = \mathbb{E}(X_{s+k} X_{t+k}), \forall s, t \in \mathbb{T}, \forall k \geq 0. \quad (2.6)$$

Since

$$\text{Cov}(X_s, X_t) = \mathbb{E}(X_s X_t) - \mathbb{E}(X_s) \mathbb{E}(X_t), \quad (2.7)$$

we see that

$$\text{Cov}(X_s, X_t) = \text{Cov}(X_{s+k}, X_{t+k}), \forall s, t \in \mathbb{T}, \forall k \geq 0, \quad (2.8)$$

so the conditions (2.5) - (2.8) are equivalent to the conditions (2.5) - (2.6). The mean of  $X_t$  is constant, and the covariance between any two variables of the process only depends on the distance between the variables, not their position in the series.  $\square$

**Proposition 2.4** EXISTENCE OF AN AUTOCOVARANCE FUNCTION. *If the process  $\{X_t : t \in \mathbb{T}\}$  is second-order stationary, then there exists a function  $\gamma : \mathbb{Z} \rightarrow \mathbb{R}$  such that*

$$\text{Cov}(X_s, X_t) = \gamma(t - s), \quad \forall s, t \in \mathbb{T}. \quad (2.9)$$

*The function  $\gamma$  is called the autocovariance function of the process  $\{X_t : t \in \mathbb{T}\}$ , and  $\gamma_k := \gamma(k)$  the lag- $k$  autocovariance of the process  $\{X_t : t \in \mathbb{T}\}$ .*

PROOF. To show the existence of the autocovariance function in (2.9), we need to prove the following implication:

$$t_2 - s_2 = t_1 - s_1 \Rightarrow \text{Cov}(X_{s_2}, X_{t_2}) = \text{Cov}(X_{s_1}, X_{t_1}) \quad (2.10)$$

for all pairs  $(s_1, t_1)$  and  $(s_2, t_2)$ . Suppose that  $t_2 - s_2 = t_1 - s_1$ . Then, using the stationarity assumption,

$$\begin{aligned} \text{Cov}(X_{s_2}, X_{t_2}) &= \text{Cov}(X_{s_2+(s_1-s_2)}, X_{t_2+(s_1-s_2)}) \\ &= \text{Cov}(X_{s_1}, X_{s_1+(t_2-s_2)}) \\ &= \text{Cov}(X_{s_1}, X_{s_1+(t_1-s_1)}) \\ &= \text{Cov}(X_{s_1}, X_{t_1}). \end{aligned} \quad (2.11)$$

□

**Proposition 2.5** PROPERTIES OF THE AUTOCOVARANCE FUNCTION. Let  $\{X_t : t \in \mathbb{T}\}$  be a second-order stationary process. The autocovariance function  $\gamma(k)$  of the process  $\{X_t : t \in \mathbb{T}\}$  satisfies the following properties:

- (1)  $\gamma(0) = V(X_t) \geq 0, \forall t \in \mathbb{T}$ ;
- (2)  $\gamma(k) = \gamma(-k), \forall k \in \mathbb{Z}$  (i.e.,  $\gamma(k)$  is an even function of  $k$ );
- (3)  $|\gamma(k)| \leq \gamma(0), \forall k \in \mathbb{Z}$ ;
- (4) the function  $\gamma(k)$  is positive semidefinite, i.e.

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j \gamma(t_i - t_j) \geq 0, \quad (2.12)$$

- for any positive integer  $N$  and for all the vectors  $a = (a_1, \dots, a_N)' \in \mathbb{R}^N$  and  $(t_1, \dots, t_N)' \in \mathbb{T}^N$ ;
- (5) any  $N \times N$  matrix of the form

$$\begin{aligned} \Gamma_N &= [\gamma(j-i)]_{i,j=1,\dots,N} \\ &= \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(N-1) \\ \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(N-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(N-1) & \gamma(N-2) & \gamma(N-3) & \cdots & \gamma(0) \end{bmatrix} \end{aligned} \quad (2.13)$$

is positive semidefinite.

**Proposition 2.6** EXISTENCE OF AN AUTOCORRELATION FUNCTION. *If the process  $\{X_t : t \in \mathbb{T}\}$  is second-order stationary, then there exists a function  $\rho : \mathbb{Z} \rightarrow [-1, 1]$  such that*

$$\rho(t-s) = \text{Corr}(X_s, X_t) = \gamma(t-s)/\gamma(0), \forall s, t \in \mathbb{T}, \quad (2.14)$$

where  $0/0 := 1$ . *The function  $\rho$  is called the autocorrelation function of the process  $\{X_t : t \in \mathbb{T}\}$ , and  $\rho_k := \rho(k)$  the lag- $k$  autocorrelation of the process  $\{X_t : t \in \mathbb{T}\}$ .*



**Proposition 2.7** PROPERTIES OF THE AUTOCORRELATION FUNCTION. Let  $\{X_t : t \in \mathbb{T}\}$  be a second-order stationary process. The autocorrelation function  $\rho(k)$  of the process  $\{X_t : t \in \mathbb{T}\}$  satisfies the following properties:

- (1)  $\rho(0) = 1$ ;
- (2)  $\rho(k) = \rho(-k)$ ,  $\forall k \in \mathbb{Z}$ ;
- (3)  $|\rho(k)| \leq 1$ ,  $\forall k \in \mathbb{Z}$ ;
- (4) the function  $\rho(k)$  is positive semidefinite, i.e.

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j \rho(t_i - t_j) \geq 0 \quad (2.15)$$

for any positive integer  $N$  and for all the vectors  $a = (a_1, \dots, a_N)' \in \mathbb{R}^N$  and  $(t_1, \dots, t_N)' \in \mathbb{T}^N$ ;

- (5) any  $N \times N$  matrix of the form

$$R_N = \frac{1}{\gamma_0} \Gamma_N = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \dots & \rho(N-1) \\ \rho(1) & 1 & \rho(1) & \dots & \rho(N-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(N-1) & \rho(N-2) & \rho(N-3) & \dots & 1 \end{bmatrix} \quad (2.16)$$

is positive semidefinite, where  $\gamma(0) = V(X_t)$ .

**Theorem 2.8** CHARACTERIZATION OF AUTOCOVARANCE FUNCTIONS. *An even function  $\gamma : \mathbb{Z} \rightarrow \mathbb{R}$  is positive semidefinite iff  $\gamma(\cdot)$  is the autocovariance function of a second-order stationary process  $\{X_t : t \in \mathbb{Z}\}$ .*

PROOF. See Brockwell and Davis (1991, Chapter 2). □

**Corollary 2.9** CHARACTERIZATION OF AUTOCORRELATION FUNCTIONS. *An even function  $\rho : \mathbb{Z} \rightarrow [-1, 1]$  is positive semidefinite iff  $\rho$  is the autocorrelation function of a second-order stationary process  $\{X_t : t \in \mathbb{Z}\}$ .*

**Definition 2.3** DETERMINISTIC PROCESS. *Let  $\{X_t : t \in \mathbb{T}\}$  be a stochastic process,  $\mathbb{T}_1 \subseteq \mathbb{T}$  and  $I_t = \{X_s : s \leq t\}$ . We say that the process  $\{X_t : t \in \mathbb{T}\}$  is deterministic on  $\mathbb{T}_1$  iff there exists a collection of functions  $\{g_t(I_{t-1}) : t \in \mathbb{T}_1\}$  such that  $X_t = g_t(I_{t-1})$  with probability one,  $\forall t \in \mathbb{T}_1$ .*

A deterministic process can be perfectly predicted from its own past (at points where it is deterministic).

**Proposition 2.10** CRITERION FOR A DETERMINISTIC PROCESS. *Let  $\{X_t : t \in \mathbb{T}\}$  be a second-order stationary process, where  $\mathbb{T} = \{t \in \mathbb{Z} : t > n_0\}$  and  $n_0 \in \mathbb{Z} \cup \{-\infty\}$ , and let  $\gamma(k)$  its autocovariance function. If there exists an integer  $N \geq 1$  such that the matrix  $\Gamma_N$  is singular [where  $\Gamma_N$  is defined in Proposition 2.5], then the process  $\{X_t : t \in \mathbb{T}\}$  is deterministic for  $t > n_0 + N - 1$ . In particular, if  $V(X_t) = \gamma(0) = 0$ , the process is deterministic for  $t \in \mathbb{T}$ .*

For a second-order indetermistic stationary process at any  $t \in \mathbb{T}$ , all the matrices  $\Gamma_N, N \geq 1$ , are invertible.

**Definition 2.4** STATIONARITY OF ORDER  $m$ . Let  $m$  be a non-negative integer. A stochastic process  $\{X_t : t \in \mathbb{T}\}$  is stationary of order  $m$  iff

$$(1) \mathbb{E}(|X_t|^m) < \infty, \forall t \in \mathbb{T},$$

and

$$(2) \mathbb{E}[X_{t_1}^{m_1} X_{t_2}^{m_2} \cdots X_{t_n}^{m_n}] = \mathbb{E}[X_{t_1+k}^{m_1} X_{t_2+k}^{m_2} \cdots X_{t_n+k}^{m_n}]$$

for any  $k \geq 0$ , and for any nonempty subset  $\{t_1, \dots, t_n\} \in \mathbb{T}^N$  and all the non-negative integers  $m_1, \dots, m_n$  such that  $m_1 + m_2 + \dots + m_n \leq m$ .

If  $m = 1$ , the mean is constant, but not necessarily the other moments. If  $m = 2$ , the process is second-order stationary.

**Definition 2.5** ASYMPTOTIC STATIONARITY OF ORDER  $m$ . Let  $m$  a non-negative integer. A stochastic process  $\{X_t : t \in \mathbb{T}\}$  is asymptotically stationary of order  $m$  iff

$$(1) \text{ there exists an integer } N \text{ such that } (|X_t|^m) < \infty, \text{ for } t \geq N,$$

and

$$(2) \lim_{t_1 \rightarrow \infty} [\mathbb{E}(X_{t_1}^{m_1} X_{t_1+\Delta_2}^{m_2} \cdots X_{t_1+\Delta_n}^{m_n}) - \mathbb{E}(X_{t_1+k}^{m_1} X_{t_1+\Delta_2+k}^{m_2} \cdots X_{t_1+\Delta_n+k}^{m_n})] = 0$$

for any  $k \geq 0$ ,  $t_1 \in \mathbb{T}$ , all the positive integers  $\Delta_2, \Delta_3, \dots, \Delta_n$  such that  $\Delta_2 < \Delta_3 < \dots < \Delta_n$ , and all non-negative integers  $m_1, \dots, m_n$  such that  $m_1 + m_2 + \dots + m_n \leq m$ .

### 3. Some important models

In this section, we will again assume that  $\mathbb{T}$  is a right-infinite interval of integers (Assumption 2.1):

$$\mathbb{T} = \{t \in \mathbb{Z} : t > n_0\} \quad \text{where } n_0 \in \mathbb{Z} \cup \{-\infty\}. \quad (3.1)$$

#### 3.1. Noise models

**Definition 3.1** SEQUENCE OF INDEPENDENT RANDOM VARIABLES. A process  $\{X_t : t \in \mathbb{T}\}$  is a sequence of independent random variables iff the variables  $X_t$  are mutually independent. This is denoted by:

$$\{X_t : t \in \mathbb{T}\} \sim IND \text{ or } \{X_t\} \sim IND. \quad (3.2)$$

Further, we write:

$$\begin{aligned} \{X_t : t \in \mathbb{T}\} &\sim IND(\mu_t) \quad \text{if } \mathbb{E}(X_t) = \mu_t, \\ \{X_t : t \in \mathbb{T}\} &\sim IND(\mu_t, \sigma_t^2) \quad \text{if } \mathbb{E}(X_t) = \mu_t \text{ and } V(X_t) = \sigma_t^2. \end{aligned} \quad (3.3)$$

**Definition 3.2** RANDOM SAMPLE. A random sample is a sequence of independent and identically distributed (i.i.d.) random variables. This is denoted by

$$\{X_t : t \in \mathbb{T}\} \sim IID. \quad (3.4)$$

A random sample is a SS process. If  $\mathbb{E}(X_t^2) < \infty$ , for all  $t \in \mathbb{T}$ , the process is S2. In this case, we write:

$$\{X_t : t \in \mathbb{T}\} \sim IID(\mu, \sigma^2), \quad \text{if } \mathbb{E}(X_t) = \mu \text{ and } V(X_t) = \sigma^2. \quad (3.5)$$

**Definition 3.3** WHITE NOISE. A white noise is a sequence of random variables in  $L_2$  with mean zero, the same variance and mutually uncorrelated, i.e.

$$\mathbb{E}(X_t^2) < \infty, \forall t \in \mathbb{T}, \quad (3.6)$$

$$\mathbb{E}(X_t^2) = \sigma^2, \forall t \in \mathbb{T}, \quad (3.7)$$

$$\text{Cov}(X_s, X_t) = 0, \text{ if } s \neq t. \quad (3.8)$$

This is denoted by:

$$\{X_t : t \in \mathbb{T}\} \sim \text{WN}(0, \sigma^2) \text{ or } \{X_t\} \sim \text{WN}(0, \sigma^2). \quad (3.9)$$

**Definition 3.4** HETEROSKEDASTIC WHITE NOISE. A heteroskedastic white noise is a sequence of random variables in  $L_2$  with mean zero and mutually uncorrelated, i.e.

$$\mathbb{E}(X_t^2) < \infty, \forall t \in \mathbb{T}, \quad (3.10)$$

$$\mathbb{E}(X_t) = 0, \forall t \in \mathbb{T}, \quad (3.11)$$

$$\text{Cov}(X_t, X_s) = 0, \text{ if } s \neq t, \quad (3.12)$$

$$\mathbb{E}(X_t^2) = \sigma_t^2, \forall t \in \mathbb{T}. \quad (3.13)$$

This is denoted by:

$$\{X_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma_t^2) \text{ or } \{X_t\} \sim \text{WN}(0, \sigma_t^2). \quad (3.14)$$

Each one of these four models will be called a *noise process*.

### 3.2. Harmonic processes

Many time series exhibit apparent periodic behavior. This suggests one to use periodic functions to describe them.

**Definition 3.5** PERIODIC FUNCTION. A function  $f(t)$ ,  $t \in \mathbb{R}$ , is periodic of period  $P$  on  $\mathbb{R}$  iff

$$f(t + P) = f(t), \quad \forall t, \quad (3.15)$$

and  $P$  is the lowest number such that (3.15) holds for all  $t$ .  $\frac{1}{P}$  is the frequency associated with the function (number of cycles per unit of time).

**Example 3.1** Sinus function:

$$\sin(t) = \sin(t + 2\pi) = \sin(t + 2\pi k), \quad \forall k \in \mathbb{Z}. \quad (3.16)$$

For the sinus function, the period is  $P = 2\pi$  and the frequency is  $f = 1 / (2\pi)$ .

**Example 3.2** Cosine function:

$$\cos(t) = \cos(t + 2\pi) = \cos(t + 2\pi k), \quad \forall k \in \mathbb{Z}. \quad (3.17)$$

**Example 3.3**

$$\sin(vt) = \sin \left[ v \left( t + \frac{2\pi}{v} \right) \right] = \sin \left[ v \left( t + \frac{2\pi k}{v} \right) \right], \quad \forall k \in \mathbb{Z}. \quad (3.18)$$

**Example 3.4**

$$\cos(vt) = \cos \left[ v \left( t + \frac{2\pi}{v} \right) \right] = \cos \left[ v \left( t + \frac{2\pi k}{v} \right) \right], \forall k \in \mathbb{Z}. \quad (3.19)$$

For  $\sin(vt)$  and  $\cos(vt)$ , the period is  $P = 2\pi/v$ .

**Example 3.5** GENERAL COSINE FUNCTION.

$$\begin{aligned} f(t) &= C \cos(vt + \theta) = C[\cos(vt) \cos(\theta) - \sin(vt) \sin(\theta)] \\ &= A \cos(vt) + B \sin(vt) \end{aligned} \tag{3.20}$$

where  $C \geq 0$ ,  $A = C \cos(\theta)$  and  $B = -C \sin \theta$ . Further,

$$C = \sqrt{A^2 + B^2}, \quad \tan(\theta) = -B/A \text{ (if } C \neq 0\text{)}.$$
 \tag{3.21}

In the above function, the different parameters have the following names:

- $C$  = amplitude;
- $v$  = angular frequency (radians/time unit);
- $P = 2\pi/v$  = period;
- $\bar{v} = \frac{1}{P} = \frac{v}{2\pi}$  = frequency (number of cycles per time unit);
- $\theta$  = phase angle (usually  $0 \leq \theta < 2\pi$  or  $-\pi/2 < \theta \leq \pi/2$ ).



### Example 3.6

$$\begin{aligned} f(t) &= C \sin(\nu t + \theta) = C \cos(\nu t + \theta - \pi/2) \\ &= C [\sin(\nu t) \cos(\theta) + \cos(\nu t) \sin(\theta)] \\ &= A \cos(\nu t) + B \sin(\nu t) \end{aligned} \tag{3.22}$$

where

$$0 \leq \nu < 2\pi, \tag{3.23}$$

$$A = C \sin(\theta) = C \cos\left(\theta - \frac{\pi}{2}\right), \tag{3.24}$$

$$B = C \cos(\theta) = -C \sin\left(\theta - \frac{\pi}{2}\right). \tag{3.25}$$

Consider the model

$$\begin{aligned} X_t &= C \cos(\mathbf{v}t + \theta) \\ &= A \cos(\mathbf{v}t) + B \sin(\mathbf{v}t), \quad t \in \mathbb{Z}. \end{aligned} \quad (3.26)$$

If  $A$  and  $B$  are constants,

$$\mathbb{E}(X_t) = A \cos(\mathbf{v}t) + B \sin(\mathbf{v}t), \quad t \in \mathbb{Z}, \quad (3.27)$$

so the process  $X_t$  is non-stationary (since the mean is not constant). Suppose now that  $A$  and  $B$  are random variables such that

$$\mathbb{E}(A) = \mathbb{E}(B) = 0, \quad \mathbb{E}(A^2) = \mathbb{E}(B^2) = \sigma^2, \quad \mathbb{E}(AB) = 0. \quad (3.28)$$

$A$  and  $B$  do not depend on  $t$  but are fixed for each realization of the process [ $A = A(\omega)$ ,  $B = B(\omega)$ ]. In this case,

$$\mathbb{E}(X_t) = 0, \quad (3.29)$$

$$\begin{aligned} \mathbb{E}(X_s X_t) &= \mathbb{E}(A^2) \cos(\mathbf{v}s) \cos(\mathbf{v}t) + \mathbb{E}(B^2) \sin(\mathbf{v}s) \sin(\mathbf{v}t) \\ &= \sigma^2 [\cos(\mathbf{v}s) \cos(\mathbf{v}t) + \sin(\mathbf{v}s) \sin(\mathbf{v}t)] \\ &= \sigma^2 \cos[\mathbf{v}(t - s)]. \end{aligned} \quad (3.30)$$

The process  $X_t$  is stationary of order 2 with the following autocovariance and autocorrelation functions:

$$\gamma_X(k) = \sigma^2 \cos(\mathbf{v}k), \quad (3.31)$$

$$\rho_X(k) = \cos(\mathbf{v}k). \quad (3.32)$$

If we add  $m$  cyclic processes of the form (3.26), we obtain a harmonic process of order  $m$ .

**Definition 3.6** HARMONIC PROCESS OF ORDER  $m$ . We say the process  $\{X_t : t \in \mathbb{T}\}$  is a harmonic process of order  $m$  if it can be written in the form

$$X_t = \sum_{j=1}^m [A_j \cos(\mathbf{v}_j t) + B_j \sin(\mathbf{v}_j t)], \quad \forall t \in \mathbb{T}, \quad (3.33)$$

where  $\mathbf{v}_1, \dots, \mathbf{v}_m$  are distinct constants in the interval  $[0, 2\pi)$ .

If  $A_j, B_j, j = 1, \dots, m$ , are random variables in  $L_2$  such that

$$\mathbb{E}(A_j) = \mathbb{E}(B_j) = 0, \quad j = 1, \dots, m, \quad (3.34)$$

$$\mathbb{E}(A_j^2) = \mathbb{E}(B_j^2) = \sigma_j^2, \quad j = 1, \dots, m, \quad (3.35)$$

$$\mathbb{E}(A_j A_k) = \mathbb{E}(B_j B_k) = 0, \quad \text{for } j \neq k, \quad (3.36)$$

$$\mathbb{E}(A_j B_k) = 0, \quad \forall j, k, \quad (3.37)$$

the harmonic process  $X_t$  is second-order stationary, with:

$$\mathbb{E}(X_t) = 0, \quad (3.38)$$

$$\mathbb{E}(X_s X_t) = \sum_{j=1}^m \sigma_j^2 \cos[\mathbf{v}_j(t-s)], \quad (3.39)$$

hence

$$\gamma_X(k) = \sum_{j=1}^m \sigma_j^2 \cos(\mathbf{v}_j k), \quad (3.40)$$

$$\rho_X(k) = \sum_{j=1}^m \sigma_j^2 \cos(\mathbf{v}_j k) / \sum_{j=1}^m \sigma_j^2. \quad (3.41)$$

If we add a white noise  $u_t$  to  $X_t$  in (3.33), we obtain again a second-order stationary process:

$$X_t = \sum_{j=1}^m [A_j \cos(\mathbf{v}_j t) + B_j \sin(\mathbf{v}_j t)] + u_t, \quad t \in \mathbb{T}, \quad (3.42)$$

where the process  $\{u_t : t \in \mathbb{T}\} \sim \text{WN}(0, \sigma^2)$  is uncorrelated with  $A_j, B_j, j = 1, \dots, m$ . In this case,  $\mathbb{E}(X_t) = 0$  and

$$\gamma_X(k) = \sum_{j=1}^m \sigma_j^2 \cos(\mathbf{v}_j k) + \sigma^2 \delta(k) \quad (3.43)$$

where

$$\begin{aligned} \delta(k) &= 1 \quad \text{if } k = 0 \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (3.44)$$

If a series can be described by an equation of the form (3.42), we can view it as a realization of a second-order stationary process.

### 3.3. Linear processes

Many stochastic processes with dependence are obtained through transformations of noise processes.

**Definition 3.7** AUTOREGRESSIVE PROCESS. *The process  $\{X_t : t \in \mathbb{T}\}$  is an autoregressive process of order  $p$  if it satisfies an equation of the form*

$$X_t = \bar{\mu} + \sum_{j=1}^p \varphi_j X_{t-j} + u_t, \quad \forall t \in \mathbb{T}, \quad (3.45)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ . In this case, we denote

$$\{X_t : t \in \mathbb{T}\} \sim \text{AR}(p). \quad (3.46)$$

Usually,  $\mathbb{T} = \mathbb{Z}$  or  $\mathbb{T} = \mathbb{Z}_+$  (positive integers). If  $\sum_{j=1}^p \varphi_j \neq 1$ , we can define

$$\mu := \bar{\mu} / (1 - \sum_{j=1}^p \varphi_j) \quad (3.47)$$

and write

$$\tilde{X}_t = \sum_{j=1}^p \varphi_j \tilde{X}_{t-j} + u_t, \quad \forall t \in \mathbb{T}, \quad (3.48)$$

where  $\tilde{X}_t := X_t - \mu$ .

**Definition 3.8** MOVING AVERAGE PROCESS. *The process  $\{X_t : t \in \mathbb{T}\}$  is a moving average process of order  $q$  if it can be written in the form*

$$X_t = \bar{\mu} + \sum_{j=0}^q \psi_j u_{t-j}, \quad \forall t \in \mathbb{T}, \quad (3.49)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ . In this case, we denote

$$\{X_t : t \in \mathbb{T}\} \sim \text{MA}(q). \quad (3.50)$$

Without loss of generality, we can set  $\psi_0 = 1$  and  $\psi_j = -\theta_j$ ,  $j = 1, \dots, q$ :

$$X_t = \bar{\mu} + u_t - \sum_{j=1}^q \theta_j u_{t-j}, \quad t \in \mathbb{T} \quad (3.51)$$

or, equivalently,

$$\tilde{X}_t = u_t - \sum_{j=1}^q \theta_j u_{t-j} \quad (3.52)$$

where  $\tilde{X}_t := X_t - \bar{\mu}$ .

**Definition 3.9** AUTOREGRESSIVE-MOVING-AVERAGE PROCESS. The process  $\{X_t : t \in \mathbb{T}\}$  is an autoregressive-moving-average (ARMA) process of order  $(p, q)$  if it can be written in the form

$$X_t = \bar{\mu} + \sum_{j=1}^p \varphi_j X_{t-j} + u_t - \sum_{j=1}^q \theta_j u_{t-j}, \quad \forall t \in \mathbb{T}, \quad (3.53)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ . In this case, we denote

$$\{X_t : t \in \mathbb{T}\} \sim \text{ARMA}(p, q). \quad (3.54)$$

If  $\sum_{j=1}^p \varphi_j \neq 1$ , we can also write

$$\tilde{X}_t = \sum_{j=1}^p \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^q \theta_j u_{t-j} \quad (3.55)$$

where  $\tilde{X}_t = X_t - \mu$  and  $\mu = \bar{\mu} / (1 - \sum_{j=1}^p \varphi_j)$ .

**Definition 3.10** MOVING AVERAGE PROCESS OF INFINITE ORDER. *The process  $\{X_t : t \in \mathbb{T}\}$  is a moving-average process of infinite order if it can be written in the form*

$$X_t = \bar{\mu} + \sum_{j=-\infty}^{+\infty} \psi_j u_{t-j}, \quad \forall t \in \mathbb{Z}, \quad (3.56)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ . We also say that  $X_t$  is a weakly linear process. In this case, we denote

$$\{X_t : t \in \mathbb{T}\} \sim \text{MA}(\infty). \quad (3.57)$$

In particular, if  $\psi_j = 0$  for  $j < 0$ , i.e.

$$X_t = \bar{\mu} + \sum_{j=0}^{\infty} \psi_j u_{t-j}, \quad \forall t \in \mathbb{Z}, \quad (3.58)$$

we say that  $X_t$  is a causal function of  $u_t$  (causal linear process).

**Definition 3.11** AUTOREGRESSIVE PROCESS OF INFINITE ORDER. *The process  $\{X_t : t \in \mathbb{T}\}$  is an autoregressive process of infinite order if it can be written in the form*

$$X_t = \bar{\mu} + \sum_{j=1}^{\infty} \varphi_j X_{t-j} + u_t, \quad t \in \mathbb{T}, \quad (3.59)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ . In this case, we denote

$$\{X_t : t \in \mathbb{T}\} \sim \text{AR}(\infty). \quad (3.60)$$



**Definition 3.12 Remark 3.1** We can generalize the notions defined above by assuming that  $\{u_t : t \in \mathbb{Z}\}$  is a noise. Unless stated otherwise, we will suppose  $\{u_t\}$  is a white noise.

**QUESTIONS :**

- (1) Under which conditions are the processes defined above stationary (strictly or in  $L_r$ )?
- (2) Under which conditions are the processes  $MA(\infty)$  or  $AR(\infty)$  well defined (convergent series)?
- (3) What are the links between the different classes of processes defined above?
- (4) When a process is stationary, what are its autocovariance and autocorrelation functions?

### 3.4. Integrated processes

**Definition 3.13** RANDOM WALK. *The process  $\{X_t : t \in \mathbb{T}\}$  is a random walk if it satisfies an equation of the form*

$$X_t - X_{t-1} = v_t, \quad \forall t \in \mathbb{T}, \quad (3.61)$$

where  $\{v_t : t \in \mathbb{T}\} \sim \text{IID}$ . To ensure that this process is well defined, we suppose that  $n_0 \neq -\infty$ . If  $n_0 = -1$ , we can write

$$X_t = X_0 + \sum_{j=1}^t v_j \quad (3.62)$$

hence the name “integrated process”. If  $\mathbb{E}(v_t) = \bar{\mu}$  or  $\text{Med}(v_t) = \bar{\mu}$ , one often writes

$$X_t - X_{t-1} = \bar{\mu} + u_t \quad (3.63)$$

where  $u_t := v_t - \bar{\mu} \sim \text{IID}$  and  $\mathbb{E}(u_t) = 0$  or  $\text{Med}(u_t) = 0$  (depending on whether  $\mathbb{E}(u_t) = 0$  or  $\text{Med}(u_t) = 0$ ). If  $\bar{\mu} \neq 0$ , we say the random walk has a drift.

**Definition 3.14** WEAK RANDOM WALK. *The process  $\{X_t : t \in \mathbb{T}\}$  is a weak random walk if  $X_t$  satisfies an equation of the form*

$$X_t - X_{t-1} = \bar{\mu} + u_t \quad (3.64)$$

where  $\{u_t : t \in \mathbb{T}\} \sim \text{WN}(0, \sigma^2)$ ,  $\{u_t : t \in \mathbb{T}\} \sim \text{WN}(0, \sigma_t^2)$ , or  $\{u_t : t \in \mathbb{T}\} \sim \text{IND}(0)$ .

**Definition 3.15** INTEGRATED PROCESS. *The process  $\{X_t : t \in \mathbb{T}\}$  is integrated of order  $d$  if it can be written in the form*

$$(1 - B)^d X_t = Z_t, \forall t \in \mathbb{T}, \quad (3.65)$$

where  $\{Z_t : t \in \mathbb{T}\}$  is a stationary process (usually stationary of order 2) and  $d$  is a non-negative integer ( $d = 0, 1, 2, \dots$ ). In particular, if  $\{Z_t : t \in \mathbb{T}\}$  is an ARMA( $p, q$ ) stationary process,  $\{X_t : t \in \mathbb{T}\}$  is an ARIMA( $p, d, q$ ) process:  $\{X_t : t \in \mathbb{T}\} \sim \text{ARIMA}(p, d, q)$ . We note

$$B X_t = X_{t-1}, \quad (3.66)$$

$$(1 - B)X_t = X_t - X_{t-1}, \quad (3.67)$$

$$(1 - B)^2 X_t = (1 - B)(1 - B)X_t = (1 - B)(X_t - X_{t-1}) \quad (3.68)$$

$$= X_t - 2X_{t-1} + X_{t-2}, \quad (3.69)$$

$$(1 - B)^d X_t = (1 - B)(1 - B)^{d-1} X_t, d = 1, 2, \dots \quad (3.70)$$

where  $(1 - B)^0 = 1$ .

### 3.5. Deterministic trends

**Definition 3.16** DETERMINISTIC TREND. *The process  $\{X_t : t \in \mathbb{T}\}$  follows a deterministic trend if it can be written in the form*

$$X_t = f(t) + Z_t, \forall t \in \mathbb{T}, \quad (3.71)$$

where  $f(t)$  is a deterministic function of time and  $\{Z_t : t \in \mathbb{T}\}$  is a noise or a stationary process.

**Example 3.7** Important cases of deterministic trend:

$$X_t = \beta_0 + \beta_1 t + u_t, \quad (3.72)$$

$$X_t = \sum_{j=0}^k \beta_j t^j + u_t, \quad (3.73)$$

where  $\{u_t : t \in \mathbb{T}\} \sim \text{WN}(0, \sigma^2)$ .

#### 4. Transformations of stationary processes

**Theorem 4.1** ABSOLUTE MOMENT SUMMABILITY CRITERION FOR CONVERGENCE OF A LINEAR TRANSFORMATION OF A STOCHASTIC PROCESS. Let  $\{X_t : t \in \mathbb{Z}\}$  be a stochastic process on the integers,  $r \geq 1$  and  $\{a_j : j \in \mathbb{Z}\}$  a sequence of real numbers. If

$$\sum_{j=-\infty}^{\infty} |a_j| \mathbb{E}(|X_{t-j}|^r)^{1/r} < \infty \quad (4.1)$$

then, for any  $t$ , the random series  $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$  converges absolutely a.s. and in mean of order  $r$  to a random variable  $Y_t$  such that  $\mathbb{E}(|Y_t|^r) < \infty$ .

PROOF. See Dufour (2016). □

**Theorem 4.2** ABSOLUTE SUMMABILITY CRITERION FOR CONVERGENCE OF A LINEAR TRANSFORMATION OF A WEAKLY STATIONARY PROCESS. Let  $\{X_t : t \in \mathbb{Z}\}$  be a second-order stationary process and  $\{a_j : j \in \mathbb{Z}\}$  an sequence of real numbers absolutely convergent sequence of real numbers, i.e.

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty. \quad (4.2)$$

Then the random series  $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$  converges absolutely a.s. and in mean of order 2 to a random variable  $Y_t \in L_2$ ,  $\forall t$ , and the process  $\{Y_t : t \in \mathbb{Z}\}$  is second-order stationary with autocovariance function

$$\gamma_Y(k) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_i a_j \gamma_X(k - i + j). \quad (4.3)$$

PROOF. See Gouriéroux and Monfort (1997, Property 5.6).  $\square$

**Theorem 4.3** NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE OF LINEAR FILTERS OF ARBITRARY WEAKLY STATIONARY PROCESSES. The series  $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$  converges absolutely a.s. for any second-order stationary process  $\{X_t : t \in \mathbb{Z}\}$  iff

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty. \quad (4.4)$$

## 5. Infinite order moving averages

We study here random series of the form

$$\sum_{j=0}^{\infty} \psi_j u_{t-j}, \quad t \in \mathbb{Z} \quad (5.1)$$

and

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, \quad t \in \mathbb{Z} \quad (5.2)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ .

### 5.1. Convergence conditions

**Theorem 5.1** MEAN SQUARE CONVERGENCE OF AN INFINITE MOVING AVERAGE. *Let  $\{\psi_j : j \in \mathbb{Z}\}$  be a sequence of fixed real constants and  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ .*

- (1) *If  $\sum_{j=1}^{\infty} \psi_j^2 < \infty$ ,  $\sum_{j=1}^{\infty} \psi_j u_{t-j}$  converges in q.m. to a random variable  $X_{U_t}$  in  $L_2$ .*
- (2) *If  $\sum_{j=-\infty}^0 \psi_j^2 < \infty$ ,  $\sum_{j=-\infty}^0 \psi_j u_{t-j}$  converges in q.m. to a random variable  $X_{L_t}$  in  $L_2$ .*
- (3) *If  $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ ,  $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$  converges in q.m. to a random variable  $X_t$  in  $L_2$ , and  $\sum_{j=-n}^n \psi_j u_{t-j} \xrightarrow[n \rightarrow \infty]{2} X_t$ .*

PROOF. Suppose  $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ . We can write

$$\sum_{j=1}^{\infty} \psi_j u_{t-j} = \sum_{j=1}^{\infty} Y_j(t), \quad \sum_{j=-\infty}^0 \psi_j u_{t-j} = \sum_{j=-\infty}^0 Y_j(t) \quad (5.3)$$

where  $Y_j(t) := \psi_j u_{t-j}$ ,

$$\mathbb{E}[Y_j(t)^2] = \psi_j^2 \mathbb{E}(u_{t-j}^2) = \psi_j^2 \sigma^2 < \infty, \text{ for } t \in \mathbb{Z},$$

and the variables  $Y_j(t)$ ,  $t \in \mathbb{Z}$ , are orthogonal. If  $\sum_{j=1}^{\infty} \psi_j^2 < \infty$ , the series  $\sum_{j=1}^{\infty} Y_j(t)$  converges in q.m. to a random variable  $X_{Ut}$  such that  $\mathbb{E}[X_{Ut}^2] < \infty$ , i.e.

$$\sum_{j=1}^n Y_j(t) \xrightarrow[n \rightarrow \infty]{2} X_{Ut} := \sum_{j=1}^{\infty} \psi_j u_{t-j}; \quad (5.4)$$

see Dufour (2016, Section on ‘‘Series of orthogonal variables’’). By a similar argument, if  $\sum_{j=-\infty}^0 \psi_j^2 < \infty$ , the

series  $\sum_{j=-\infty}^0 Y_j(t)$  converges in q.m. to a random variable  $X_{Lt}$  such that  $\mathbb{E}[X_{Lt}^2] < \infty$ , i.e.

$$\sum_{j=-m}^0 Y_j(t) \xrightarrow[m \rightarrow \infty]{2} X_{Lt} := \sum_{j=-\infty}^0 \psi_j u_{t-j}. \quad (5.5)$$



Finally, if  $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ , we must have  $\sum_{j=1}^{\infty} \psi_j^2 < \infty$  and  $\sum_{j=-\infty}^0 \psi_j^2 < \infty$ , hence

$$\sum_{j=-m}^n Y_j(t) = \sum_{j=-m}^0 Y_j(t) + \sum_{j=1}^n Y_j(t) \xrightarrow[m \rightarrow \infty]{n \rightarrow \infty} X_{L_t} + X_{U_t} := X_t := \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \quad (5.6)$$

where, by the  $c_r$ -inequality [see Dufour (2022)],

$$\mathbb{E}[X_t^2] = \mathbb{E}[(X_{L_t} + X_{U_t})^2] \leq 2\{\mathbb{E}[X_{L_t}^2] + \mathbb{E}[X_{U_t}^2]\} < \infty. \quad (5.7)$$

The random variable  $X_t$  is denoted:

$$X_t := \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}. \quad (5.8)$$

The last statement on the convergence of  $\sum_{j=-n}^n \psi_j u_{t-j}$  follows from the definition of mean-square convergence of  $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ . □

**Corollary 5.2** ALMOST SURE CONVERGENCE OF AN INFINITE MOVING AVERAGE. Let  $\{\psi_j : j \in \mathbb{Z}\}$  be a sequence of fixed real constants, and  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ .

(1) If  $\sum_{j=1}^{\infty} |\psi_j| < \infty$ ,  $\sum_{j=1}^{\infty} \psi_j u_{t-j}$  converges a.s. and in q.m. to a random variable  $X_{U_t}$  in  $L_2$ .

(2) If  $\sum_{j=-\infty}^0 |\psi_j| < \infty$ ,  $\sum_{j=-\infty}^0 \psi_j u_{t-j}$  converges a.s. and in q.m. to a random variable  $X_{L_t}$  in  $L_2$ .

(3) If  $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ ,  $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$  converges a.s. and in q.m. to a random variable  $X_t$  in  $L_2$ ,  $\sum_{j=-n}^n \psi_j u_{t-j} \xrightarrow[n \rightarrow \infty]{a.s.} X_t$   
and  $\sum_{j=-n}^n \psi_j u_{t-j} \xrightarrow[n \rightarrow \infty]{2} X_t$ .

PROOF. This result from Theorem 5.1 and the observation that

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty. \quad (5.9)$$

□

**Theorem 5.3** ALMOST SURE CONVERGENCE OF AN INFINITE MOVING AVERAGE OF INDEPENDENT VARIABLES. Let  $\{\psi_j : j \in \mathbb{Z}\}$  be a sequence of fixed real constants, and  $\{u_t : t \in \mathbb{Z}\} \sim \text{IID}(0, \sigma^2)$ .

(1) If  $\sum_{j=1}^{\infty} \psi_j^2 < \infty$ ,  $\sum_{j=1}^{\infty} \psi_j u_{t-j}$  converges a.s. and in q.m. to a random variable  $X_{U_t}$  in  $L_2$ .

(2) If  $\sum_{j=-\infty}^0 \psi_j^2 < \infty$ ,  $\sum_{j=-\infty}^0 \psi_j u_{t-j}$  converges a.s. and in q.m. to a random variable  $X_{L_t}$  in  $L_2$ .

(3) If  $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ ,  $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$  converges a.s. and in q.m. to a random variable  $X_t$  in  $L_2$ ,  $\sum_{j=-n}^n \psi_j u_{t-j} \xrightarrow[n \rightarrow \infty]{a.s.} X_t$   
and  $\sum_{j=-n}^n \psi_j u_{t-j} \xrightarrow[n \rightarrow \infty]{2} X_t$ .

PROOF. This result from Theorem 5.1 and by applying results on the convergence of series of independent variables [Dufour (2016, Section on “Series of independent variables”)].  $\square$

## 5.2. Mean, variance and covariances

Let

$$S_n(t) = \sum_{j=-n}^n \Psi_j u_{t-j}. \quad (5.10)$$

By Theorem 5.1, we have:

$$S_n(t) \xrightarrow[n \rightarrow \infty]{2} X_t \quad (5.11)$$

where  $X_t \in L_2$ , hence [using Dufour (2016, Section on ‘‘Convergence of functions of random variables’’)]

$$\mathbb{E}(X_t) = \lim_{n \rightarrow \infty} \mathbb{E}[S_n(t)] = 0, \quad (5.12)$$

$$V(X_t) = \mathbb{E}(X_t^2) = \lim_{n \rightarrow \infty} \mathbb{E}[S_n(t)^2] = \lim_{n \rightarrow \infty} \sum_{j=-n}^n \Psi_j^2 \sigma^2 = \sigma^2 \sum_{j=-\infty}^{\infty} \Psi_j^2, \quad (5.13)$$

$$\begin{aligned} \text{Cov}(X_t, X_{t+k}) &= \mathbb{E}(X_t X_{t+k}) = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \sum_{i=-n}^n \Psi_i u_{t-i} \right) \left( \sum_{j=-n}^n \Psi_j u_{t+k-j} \right) \right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=-n}^n \sum_{j=-n}^n \Psi_i \Psi_j \mathbb{E}(u_{t-i} u_{t+k-j}) \\ &= \begin{cases} \lim_{n \rightarrow \infty} \sum_{i=-n}^{n-k} \Psi_i \Psi_{i+k} \sigma^2 = \sigma^2 \sum_{i=-\infty}^{\infty} \Psi_i \Psi_{i+k}, & \text{if } k \geq 1, \\ \lim_{n \rightarrow \infty} \sum_{j=-n}^n \Psi_j \Psi_{j+|k|} \sigma^2 = \sigma^2 \sum_{j=-\infty}^{\infty} \Psi_j \Psi_{j+|k|}, & \text{if } k \leq -1, \end{cases} \end{aligned} \quad (5.14)$$

since  $t - i = t + k - j \Rightarrow j = i + k$  and  $i = j - k$ . For any  $k \in \mathbb{Z}$ , we can write

$$\text{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}, \quad (5.15)$$

$$\text{Corr}(X_t, X_{t+k}) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=-\infty}^{\infty} \psi_j^2. \quad (5.16)$$

The series  $\sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k}$  converges absolutely, for

$$\left| \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} \right| \leq \sum_{j=-\infty}^{\infty} |\psi_j \psi_{j+k}| \leq \left[ \sum_{j=-\infty}^{\infty} \psi_j^2 \right]^{\frac{1}{2}} \left[ \sum_{j=-\infty}^{\infty} \psi_{j+k}^2 \right]^{\frac{1}{2}} < \infty. \quad (5.17)$$

If  $X_t = \mu + X_t = \mu + \sum_{j=-\infty}^{+\infty} \psi_j u_{t-j}$ , then

$$\mathbb{E}(X_t) = \mu, \quad \text{Cov}(X_t, X_{t+k}) = \text{Cov}(X_t, X_{t+k}). \quad (5.18)$$

In the case of a causal MA( $\infty$ ) process causal, we have

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} \quad (5.19)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ ,

$$\text{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}, \quad (5.20)$$

$$\text{Corr}(X_t, X_{t+k}) = \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=0}^{\infty} \psi_j^2. \quad (5.21)$$

### 5.3. Stationarity

The process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, \quad t \in \mathbb{Z}, \quad (5.22)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$  and  $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ , is second-order stationary, for  $\mathbb{E}(X_t)$  and  $\text{Cov}(X_t, X_{t+k})$  do not depend on  $t$ . If we suppose that  $\{u_t : t \in \mathbb{Z}\} \sim \text{IID}$ , with  $\mathbb{E}|u_t| < \infty$  and  $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ , the process is strictly stationary.

### 5.4. Operational notation

We can denote the process  $\text{MA}(\infty)$

$$X_t = \mu + \psi(B)u_t = \mu + \left( \sum_{j=-\infty}^{\infty} \psi_j B^j \right) u_t \quad (5.23)$$

where  $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$  and  $B^j u_t = u_{t-j}$ .

## 6. Finite order moving averages

The MA( $q$ ) process can be written

$$X_t = \mu + u_t - \sum_{j=1}^q \theta_j u_{t-j} \quad (6.1)$$

where  $\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$ . This process is a special case of the MA( $\infty$ ) process with

$$\begin{aligned} \psi_0 &= 1, \psi_j = -\theta_j, \text{ for } 1 \leq j \leq q, \\ \psi_j &= 0, \text{ for } j < 0 \text{ or } j > q. \end{aligned} \quad (6.2)$$

This process is clearly second-order stationary, with

$$\mathbb{E}(X_t) = \mu, \quad (6.3)$$

$$V(X_t) = \sigma^2 \left( 1 + \sum_{j=1}^q \theta_j^2 \right), \quad (6.4)$$

$$\gamma(k) : = \text{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}. \quad (6.5)$$

On defining  $\theta_0 := -1$ , we then see that

$$\gamma(k) = \sigma^2 \sum_{j=0}^{q-k} \theta_j \theta_{j+k}$$



$$= \sigma^2 \left[ -\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k} \right] \quad (6.6)$$

$$= \sigma^2 [-\theta_k + \theta_1 \theta_{k+1} + \dots + \theta_{q-k} \theta_q], \text{ for } 1 \leq k \leq q, \quad (6.7)$$

$$\begin{aligned} \gamma(k) &= 0, \text{ for } k \geq q+1, \\ \gamma(-k) &= \gamma(k), \text{ for } k < 0. \end{aligned} \quad (6.8)$$

The autocorrelation function of  $X_t$  is thus

$$\begin{aligned} \rho(k) &= \left( -\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k} \right) / \left( 1 + \sum_{j=1}^q \theta_j^2 \right), \text{ for } 1 \leq k \leq q \\ &= 0, \text{ for } k \geq q+1 \end{aligned} \quad (6.9)$$

The autocorrelations are zero for  $k \geq q+1$ .

For  $q = 1$ ,

$$\rho(k) = \begin{cases} -\theta_1/(1 + \theta_1^2), & \text{if } k = 1, \\ 0, & \text{if } k \geq 2, \end{cases} \quad (6.10)$$

hence  $|\rho(1)| \leq 0.5$ . For  $q = 2$ ,

$$\rho(k) = \begin{cases} (-\theta_1 + \theta_1\theta_2)/(1 + \theta_1^2 + \theta_2^2), & \text{if } k = 1, \\ -\theta_2/(1 + \theta_1^2 + \theta_2^2), & \text{if } k = 2, \\ 0, & \text{if } k \geq 3, \end{cases} \quad (6.11)$$

hence  $|\rho(2)| \leq 0.5$ .

For any MA( $q$ ) process,

$$\rho(q) = -\theta_q/(1 + \theta_1^2 + \dots + \theta_q^2), \quad (6.12)$$

hence  $|\rho(q)| \leq 0.5$ .

There are general constraints on the autocorrelations of an MA( $q$ ) process:

$$|\rho(k)| \leq \cos(\pi/\{[q/k] + 2\}) \quad (6.13)$$

where  $[x]$  is the largest integer less than or equal to  $x$ . From the latter formula, we see:

$$\begin{aligned} \text{for } q = 1, & \quad |\rho(1)| \leq \cos(\pi/3) = 0.5, \\ \text{for } q = 2, & \quad |\rho(1)| \leq \cos(\pi/4) = 0.7071, \\ & \quad |\rho(2)| \leq \cos(\pi/3) = 0.5, \\ \text{for } q = 3, & \quad |\rho(1)| \leq \cos(\pi/5) = 0.809, \\ & \quad |\rho(2)| \leq \cos(\pi/3) = 0.5, \\ & \quad |\rho(3)| \leq \cos(\pi/3) = 0.5. \end{aligned} \quad (6.14)$$

See Chanda (1962), and Kendall, Stuart, and Ord (1983, p. 519).

## 7. Autoregressive processes

Consider a process  $\{X_t : t \in \mathbb{Z}\}$  which satisfies the equation:

$$X_t = \bar{\mu} + \sum_{j=1}^p \varphi_j X_{t-j} + u_t, \quad \forall t \in \mathbb{Z}, \quad (7.1)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ . In symbolic notation,

$$\varphi(B)X_t = \bar{\mu} + u_t, \quad t \in \mathbb{Z}, \quad (7.2)$$

where  $\varphi(B) = 1 - \varphi_1 B - \dots - \varphi_p B^p$ .

### 7.1. Stationarity

Consider the process AR(1)

$$X_t = \varphi_1 X_{t-1} + u_t, \varphi_1 \neq 0. \quad (7.3)$$

If  $X_t$  is S2,

$$\mathbb{E}(X_t) = \varphi_1 \mathbb{E}(X_{t-1}) = \varphi_1 \mathbb{E}(X_t) \quad (7.4)$$

hence  $\mathbb{E}(X_t) = 0$ . By successive substitutions,

$$\begin{aligned} X_t &= \varphi_1 [\varphi_1 X_{t-2} + u_{t-1}] + u_t \\ &= u_t + \varphi_1 u_{t-1} + \varphi_1^2 X_{t-2} \\ &= \sum_{j=0}^{N-1} \varphi_1^j u_{t-j} + \varphi_1^N X_{t-N}. \end{aligned} \quad (7.5)$$

If we suppose that  $X_t$  is S2 with  $\mathbb{E}(X_t^2) \neq 0$ , we see that

$$\mathbb{E} \left[ \left( X_t - \sum_{j=0}^{N-1} \varphi_1^j u_{t-j} \right)^2 \right] = \varphi_1^{2N} \mathbb{E}(X_{t-N}^2) = \varphi_1^{2N} \mathbb{E}(X_t^2) \xrightarrow{N \rightarrow \infty} 0 \Leftrightarrow |\varphi_1| < 1. \quad (7.6)$$

The series  $\sum_{j=0}^{\infty} \varphi_1^j u_{t-j}$  converges in *q.m.* to

$$X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} := (1 - \varphi_1 B)^{-1} u_t = \frac{1}{1 - \varphi_1 B} u_t \quad (7.7)$$

where

$$(1 - \varphi_1 B)^{-1} = \sum_{j=0}^{\infty} \varphi_1^j B^j. \quad (7.8)$$

Since

$$\sum_{j=0}^{\infty} \mathbb{E} |\varphi_1^j u_{t-j}| \leq \sigma \sum_{j=0}^{\infty} |\varphi_1|^j = \frac{\sigma}{1 - |\varphi_1|} < \infty \quad (7.9)$$

when  $|\varphi_1| < 1$ , the convergence is also a.s. The process  $X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j}$  is S2.

When  $|\varphi_1| < 1$ , the difference equation

$$(1 - \varphi_1 B)X_t = u_t \quad (7.10)$$

has a unique stationary solution which can be written

$$X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} = (1 - \varphi_1 B)^{-1} u_t. \quad (7.11)$$

The latter is thus a causal MA( $\infty$ ) process.

This condition is sufficient (but non necessary) for the existence of a unique stationary solution. The stationarity condition is often expressed by saying that the polynomial  $\varphi(z) = 1 - \varphi_1 z$  has all its roots outside the unit circle  $|z| = 1$ :

$$1 - \varphi_1 z_* = 0 \Leftrightarrow z_* = \frac{1}{\varphi_1} \quad (7.12)$$

where  $|z_*| = 1/|\varphi_1| > 1$ . In this case, we also have  $\mathbb{E}(X_{t-k}u_t) = 0, \forall k \geq 1$ . The same conclusion holds if we consider the general process

$$X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t. \quad (7.13)$$

For the AR( $p$ ) process,

$$X_t = \bar{\mu} + \sum_{j=1}^p \varphi_j X_{t-j} + u_t \quad (7.14)$$

or

$$\varphi(\mathbf{B})X_t = \bar{\mu} + u_t, \quad (7.15)$$

the stationarity condition is the following:

if the polynomial  $\varphi(z) = 1 - \varphi_1 z - \dots - \varphi_p z^p$  has all its roots outside the unit circle, the equation (7.14) has one and only one weakly stationary solution. (7.16)

$\varphi(z)$  is a polynomial of order  $p$  with no root equal to zero. It can be written in the form

$$\varphi(z) = (1 - G_1 z)(1 - G_2 z) \dots (1 - G_p z), \quad (7.17)$$

so the roots of  $\varphi(z)$  are

$$z_1^* = 1/G_1, \dots, z_p^* = 1/G_p, \quad (7.18)$$

and the stationarity condition have the equivalent form:

$$|G_j| < 1, \quad j = 1, \dots, p. \quad (7.19)$$



The stationary solution can be written

$$X_t = \varphi(\mathbf{B})^{-1} \bar{\mu} + \varphi(\mathbf{B})^{-1} u_t = \mu + \varphi(\mathbf{B})^{-1} u_t \quad (7.20)$$

where

$$\mu = \bar{\mu} / \left( 1 - \sum_{j=1}^p \varphi_j \right), \quad (7.21)$$

$$\begin{aligned} \varphi(\mathbf{B})^{-1} &= \prod_{j=1}^p (1 - G_j \mathbf{B})^{-1} = \prod_{j=1}^p \left( \sum_{k=0}^{\infty} G_j^k \mathbf{B}^k \right) \\ &= \sum_{j=1}^p \frac{K_j}{1 - G_j \mathbf{B}} \end{aligned} \quad (7.22)$$

and  $K_1, \dots, K_p$  are constants (expansion in partial fractions). Consequently,

$$\begin{aligned} X_t &= \mu + \sum_{j=1}^p \left( \frac{K_j}{1 - G_j \mathbf{B}} \right) u_t \\ &= \mu + \sum_{k=0}^{\infty} \psi_k u_{t-k} = \mu + \psi(\mathbf{B}) u_t \end{aligned} \quad (7.23)$$

where  $\psi_k = \sum_{j=1}^p K_j G_j^k$ . Thus

$$\mathbb{E}(X_{t-j} u_t) = 0, \quad \forall j \geq 1. \quad (7.24)$$

For the process AR(1) and AR(2), the stationarity conditions can be written as follows.

(a) AR(1) – For  $(1 - \varphi_1 B)X_t = \bar{\mu} + u_t$ , (7.25)

$$|\varphi_1| < 1$$

(b) AR(2) – For  $(1 - \varphi_1 B - \varphi_2 B^2)X_t = \bar{\mu} + u_t$ , (7.26)

$$\varphi_2 + \varphi_1 < 1$$

$$\varphi_2 - \varphi_1 < 1$$
(7.27)

$$-1 < \varphi_2 < 1$$
(7.28)

## 7.2. Mean, variance and autocovariances

Suppose:

a) the autoregressive process  $X_t$  is second-order stationary with  $\sum_{j=1}^p \varphi_j \neq 1$  (7.29)

and

b)  $\mathbb{E}(X_{t-j}u_t) = 0, \forall j \geq 1,$

*i.e.*, we assume that  $X_t$  is a weakly stationary solution of the equation (7.14) such that  $\mathbb{E}(X_{t-j}u_t) = 0, \forall j \geq 1.$

By the stationarity assumption, we have:  $\mathbb{E}(X_t) = \mu, \forall t,$  hence

$$\mu = \bar{\mu} + \sum_{j=1}^p \varphi_j \mu \quad (7.30)$$

and

$$\mathbb{E}(X_t) = \mu = \bar{\mu} / \left( 1 - \sum_{j=1}^p \varphi_j \right). \quad (7.31)$$

For stationarity to hold, it is necessary that  $\sum_{j=1}^p \varphi_j \neq 1.$  Let us rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^p \varphi_j \tilde{X}_{t-j} + u_t \quad (7.32)$$

where  $\tilde{X}_t = X_t - \mu$ ,  $\mathbb{E}(\tilde{X}_t) = 0$ . Then, for  $k \geq 0$ ,

$$\tilde{X}_{t+k} = \sum_{j=1}^p \varphi_j \tilde{X}_{t+k-j} + u_{t+k}, \quad (7.33)$$

$$\mathbb{E}(\tilde{X}_{t+k} \tilde{X}_t) = \sum_{j=1}^p \varphi_j \mathbb{E}(\tilde{X}_{t+k-j} \tilde{X}_t) + \mathbb{E}(u_{t+k} \tilde{X}_t), \quad (7.34)$$

$$\gamma(k) = \sum_{j=1}^p \varphi_j \gamma(k-j) + \mathbb{E}(u_{t+k} \tilde{X}_t), \quad (7.35)$$

where

$$\begin{aligned} \mathbb{E}(u_{t+k} \tilde{X}_t) &= \sigma^2, \quad \text{if } k = 0, \\ &= 0, \quad \text{if } k \geq 1. \end{aligned} \quad (7.36)$$

Thus

$$\rho(k) = \sum_{j=1}^p \varphi_j \rho(k-j), \quad k \geq 1. \quad (7.37)$$

These formulae are called the ‘‘Yule-Walker equations’’. If we know  $\rho(0), \dots, \rho(p-1)$ , we can easily compute  $\rho(k)$  for  $k \geq p+1$ . We can also write the Yule-Walker equations in the form:

$$\varphi(B)\rho(k) = 0, \quad \text{for } k \geq 1, \quad (7.38)$$

where  $B^j \rho(k) := \rho(k-j)$ . To obtain  $\rho(1), \dots, \rho(p-1)$  for  $p > 1$ , it is sufficient to solve the linear equation

system:

$$\begin{aligned}
\rho(1) &= \varphi_1 + \varphi_2\rho(1) + \cdots + \varphi_p\rho(p-1) \\
\rho(2) &= \varphi_1\rho(1) + \varphi_2 + \cdots + \varphi_p\rho(p-2) \\
&\vdots \\
\rho(p-1) &= \varphi_1\rho(p-2) + \varphi_2\rho(p-3) + \cdots + \varphi_p\rho(1)
\end{aligned} \tag{7.39}$$

where we use the identity  $\rho(-j) = \rho(j)$ . The other autocorrelations may then be obtained by recurrence:

$$\rho(k) = \sum_{j=1}^p \varphi_j \rho(k-j), \quad k \geq p. \tag{7.40}$$

To compute  $\gamma(0) = V(X_t)$ , we solve the equation

$$\begin{aligned}
\gamma(0) &= \sum_{j=1}^p \varphi_j \gamma(-j) + \mathbb{E}(u_t \tilde{X}_t) \\
&= \sum_{j=1}^p \varphi_j \gamma(j) + \sigma^2
\end{aligned} \tag{7.41}$$

hence, using  $\gamma(j) = \rho(j)\gamma(0)$ ,

$$\gamma(0) \left[ 1 - \sum_{j=1}^p \varphi_j \rho(j) \right] = \sigma^2 \tag{7.42}$$

and

$$\gamma(0) = \frac{\sigma^2}{1 - \sum_{j=1}^p \varphi_j \rho(j)}.$$

(7.43)

### 7.3. Special cases

#### 1. AR(1) – If

$$\tilde{X}_t = \varphi_1 \tilde{X}_{t-1} + u_t \quad (7.44)$$

we have:

$$\rho(1) = \varphi_1, \quad (7.45)$$

$$\rho(k) = \varphi_1 \rho(k-1), \text{ for } k \geq 1, \quad (7.46)$$

$$\rho(2) = \varphi_1 \rho(1) = \varphi_1^2, \quad (7.47)$$

$$\rho(k) = \varphi_1^k, k \geq 1, \quad (7.48)$$

$$\gamma(0) = V(X_t) = \frac{\sigma^2}{1 - \varphi_1^2}. \quad (7.49)$$

These is no constraint on  $\rho(1)$ , but there are constraints on  $\rho(k)$  for  $k \geq 2$ .

#### 2. AR(2) – If

$$X_t = \varphi_1 \tilde{X}_{t-1} + \varphi_2 \tilde{X}_{t-2} + u_t, \quad (7.50)$$

we have:

$$\rho(1) = \varphi_1 + \varphi_2 \rho(1), \quad (7.51)$$

$$\rho(1) = \frac{\varphi_1}{1 - \varphi_2}, \quad (7.52)$$

$$\rho(2) = \frac{\varphi_1^2}{1 - \varphi_2} + \varphi_2 = \frac{\varphi_1^2 + \varphi_2(1 - \varphi_2)}{1 - \varphi_2}, \quad (7.53)$$

$$\rho(k) = \varphi_1 \rho(k-1) + \varphi_2 \rho(k-2), \text{ for } k \geq 2. \quad (7.54)$$

Constraints on  $\rho(1)$  and  $\rho(2)$  are entailed by the stationarity of the AR(2) model:

$$|\rho(1)| < 1, |\rho(2)| < 1, \quad (7.55)$$

$$\rho(1)^2 < \frac{1}{2}[1 + \rho(2)]; \quad (7.56)$$

see Box and Jenkins (1976, p. 61).



#### 7.4. Explicit form for the autocorrelations

The autocorrelations of an AR( $p$ ) process satisfy the equation

$$\rho(k) = \sum_{j=1}^p \varphi_j \rho(k-j), k \geq 1, \quad (7.57)$$

where  $\rho(0) = 1$  and  $\rho(-k) = \rho(k)$ , or equivalently

$$\varphi(B)\rho(k) = 0, k \geq 1. \quad (7.58)$$

The autocorrelations can be obtained by solving the homogeneous difference equation (7.57).

The polynomial  $\varphi(z)$  has  $m$  distinct non-zero roots  $z_1^*, \dots, z_m^*$  (where  $1 \leq m \leq p$ ) with multiplicities  $p_1, \dots, p_m$  (where  $\sum_{j=1}^m p_j = p$ ), so that  $\varphi(z)$  can be written

$$\varphi(z) = (1 - G_1 z)^{p_1} (1 - G_2 z)^{p_2} \dots (1 - G_m z)^{p_m} \quad (7.59)$$

where  $G_j = 1/z_j^*$ ,  $j = 1, \dots, m$ . The roots are real or complex numbers. If  $z_j^*$  is a complex (non real) root, its conjugate  $\bar{z}_j^*$  is also a root. Consequently, the solutions of equation (7.57) have the general form

$$\rho(k) = \sum_{j=1}^m \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) G_j^k, k \geq 1, \quad (7.60)$$

where the  $A_{j\ell}$  are (possibly complex) constants which can be determined from the values  $p$  autocorrelations. We can easily find  $\rho(1), \dots, \rho(p)$  from the Yule-Walker equations.

If we write  $G_j = r_j e^{i\theta_j}$ , where  $i = \sqrt{-1}$  while  $r_j$  and  $\theta_j$  are real numbers ( $r_j > 0$ ), we see that

$$\begin{aligned}
 \rho(k) &= \sum_{j=1}^m \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) r_j^k e^{i\theta_j k} \\
 &= \sum_{j=1}^m \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) r_j^k [\cos(\theta_j k) + i \sin(\theta_j k)] \\
 &= \sum_{j=1}^m \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) r_j^k \cos(\theta_j k).
 \end{aligned} \tag{7.61}$$

By stationarity,  $0 < |G_j| = r_j < 1$  so that  $\rho(k) \rightarrow 0$  when  $k \rightarrow \infty$ . The autocorrelations decrease at an exponential rate with oscillations.

### 7.5. MA( $\infty$ ) representation of an AR( $p$ ) process

We have seen that a weakly stationary process which satisfies the equation

$$\varphi(\mathbf{B})\tilde{X}_t = u_t \quad (7.62)$$

where  $\varphi(\mathbf{B}) = 1 - \varphi_1\mathbf{B} - \dots - \varphi_p\mathbf{B}^p$ , can be written as

$$\tilde{X}_t = \psi(\mathbf{B})u_t \quad (7.63)$$

with

$$\psi(\mathbf{B}) = \varphi(\mathbf{B})^{-1} = \sum_{j=0}^{\infty} \psi_j \mathbf{B}^j \quad (7.64)$$

To compute the coefficients  $\psi_j$ , it is sufficient to note that

$$\varphi(\mathbf{B})\psi(\mathbf{B}) = 1. \quad (7.65)$$

Setting  $\psi_j = 0$  for  $j < 0$ , we see that

$$\begin{aligned} \left(1 - \sum_{k=1}^p \varphi_k \mathbf{B}^k\right) \left(\sum_{j=-\infty}^{\infty} \psi_j \mathbf{B}^j\right) &= \sum_{j=-\infty}^{\infty} \psi_j \left(\mathbf{B}^j - \sum_{k=1}^p \varphi_k \mathbf{B}^{j+k}\right) \\ &= \sum_{j=-\infty}^{\infty} \left(\psi_j - \sum_{k=1}^p \varphi_k \psi_{j-k}\right) \mathbf{B}^j = \sum_{j=-\infty}^{\infty} \tilde{\psi}_j \mathbf{B}^j = 1. \end{aligned} \quad (7.66)$$

Thus  $\tilde{\psi}_j = 1$ , if  $j = 0$ , and  $\tilde{\psi}_j = 0$ , if  $j \neq 0$ . Consequently,

$$\begin{aligned} \varphi(B)\psi_j &= \psi_j - \sum_{k=1}^p \varphi_k \psi_{j-k} = 1, \text{ if } j = 0 \\ &= 0, \text{ if } j \neq 0, \end{aligned} \tag{7.67}$$

where  $B^k \psi_j := \psi_{j-k}$ . Since  $\psi_j = 0$  for  $j < 0$ , we see that:

$$\begin{aligned} \psi_0 &= 1, \\ \psi_j &= \sum_{k=1}^p \varphi_k \psi_{j-k}, \text{ for } j \geq 1. \end{aligned} \tag{7.68}$$

More explicitly,

$$\begin{aligned} \psi_0 &= 1, \\ \psi_1 &= \varphi_1 \psi_0 = \varphi_1, \\ \psi_2 &= \varphi_1 \psi_1 + \varphi_2 \psi_0 = \varphi_1^2 + \varphi_2, \\ \psi_3 &= \varphi_1 \psi_2 + \varphi_2 \psi_1 + \varphi_3 = \varphi_1^3 + 2 \varphi_2 \varphi_1 + \varphi_3, \\ &\vdots \\ \psi_p &= \sum_{k=1}^p \varphi_k \psi_{j-k}, \\ \psi_j &= \sum_{k=1}^p \varphi_k \psi_{j-k}, j \geq p+1. \end{aligned} \tag{7.69}$$

Under the stationarity condition *i.e.*, the roots of  $\varphi(z) = 0$  are outside the unit circle], the coefficients  $\psi_j$  decline at an exponential rate as  $j \rightarrow \infty$ , possibly with oscillations.

Given the representation

$$\tilde{X}_t = \psi(B)u_t = \sum_{j=0}^{\infty} \psi_j u_{t-j}, \quad (7.70)$$

we can easily compute the autocovariances and autocorrelations of  $X_t$  :

$$\text{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}, \quad (7.71)$$

$$\text{Corr}(X_t, X_{t+k}) = \left( \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} \right) / \left( \sum_{j=0}^{\infty} \psi_j^2 \right). \quad (7.72)$$

However, this has the drawback of requiring one to compute limits of series.

### 7.6. Partial autocorrelations

The Yule-Walker equations allow one to determine the autocorrelations from the coefficients  $\varphi_1, \dots, \varphi_p$ . In the same way we can determine  $\varphi_1, \dots, \varphi_p$  from the autocorrelations

$$\rho(k) = \sum_{j=1}^p \varphi_j \rho(k-j), \quad k = 1, 2, 3, \dots \quad (7.73)$$

Taking into account the fact that  $\rho(0) = 1$  and  $\rho(-k) = \rho(k)$ , we see that

$$\begin{bmatrix} 1 & \rho(1) & \rho(2) & \dots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \dots & \rho(p-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \dots & 1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix} \quad (7.74)$$

or, equivalently,

$$R(p) \bar{\varphi}(p) = \bar{\rho}(p) \quad (7.75)$$

where

$$R(p) = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \dots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \dots & \rho(p-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \dots & 1 \end{bmatrix}, \quad (7.76)$$

$$\bar{\rho}(p) = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix}, \quad \bar{\varphi}(p) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{bmatrix}. \quad (7.77)$$

Consider now the sequence of equations

$$R(k) \bar{\varphi}(k) = \bar{\rho}(k), \quad k = 1, 2, 3, \dots \quad (7.78)$$

where

$$R(k) = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \dots & \rho(k-1) \\ \rho(1) & 1 & \rho(1) & \dots & \rho(k-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(k-1) & \rho(k-2) & \rho(k-3) & \dots & 1 \end{bmatrix}, \quad (7.79)$$

$$\bar{\rho}(k) = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{bmatrix}, \quad \bar{\varphi}(k) = \begin{bmatrix} \varphi(1|k) \\ \varphi(2|k) \\ \vdots \\ \varphi(k|k) \end{bmatrix}, \quad k = 1, 2, 3, \dots \quad (7.80)$$

so that we can solve for  $\bar{\varphi}(k)$ :

$$\bar{\varphi}(k) = R(k)^{-1} \bar{\rho}(k). \quad (7.81)$$

[If  $\sigma^2 > 0$ , we can show that  $R(k)^{-1}$  exists,  $\forall k \geq 1$ ]. On using (7.75), we see easily that:

$$\varphi_k(k) = 0 \text{ for } k \geq p + 1. \quad (7.82)$$

The coefficients  $\varphi_{kk}$  are called the lag-  $k$  partial autocorrelations.

In particular,

$$\varphi_1(|1) = \rho(1), \quad (7.83)$$

$$\varphi_2(2|2) = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2}, \quad (7.84)$$

$$\varphi_3(3|3) = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}}. \quad (7.85)$$



The partial autocorrelations may be computed using the following recursive formulae:

$$\varphi(k+1|k+1) = \frac{\rho(k+1) - \sum_{j=1}^k \varphi(j|k) \rho(k+1-j)}{1 - \sum_{j=1}^k \varphi(j|k) \rho(j)}, \quad (7.86)$$

$$\varphi(j|k+1) = \varphi(j|k) - \varphi(k+1|k+1) \varphi(k+1-j|k), \quad j = 1, 2, \dots, k. \quad (7.87)$$

Given  $\rho(1), \dots, \rho(k+1)$  and  $\varphi_1(k), \dots, \varphi_k(k)$ , we can compute  $\varphi_j(k+1), j = 1, \dots, k+1$ . The expressions (7.86) - (7.87) are called the *Durbin-Levinson formulae*; see Durbin (1960) and Box and Jenkins (1976, pp. 82-84).

## 8. Mixed processes

Consider a process  $\{X_t : t \in \mathbb{Z}\}$  which satisfies the equation:

$$X_t = \bar{\mu} + \sum_{j=1}^p \varphi_j X_{t-j} + u_t - \sum_{j=1}^q \theta_j u_{t-j} \quad (8.1)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ . Using operational notation, this can written

$$\varphi(B) X_t = \bar{\mu} + \theta(B) u_t. \quad (8.2)$$

### 8.1. Stationarity conditions

If the polynomial  $\varphi(z) = 1 - \varphi_1 z - \dots - \varphi_p z^p$  has all its roots outside the unit circle, the equation (8.1) has one and only one weakly stationary solution, which can be written:

$$X_t = \mu + \frac{\theta(B)}{\varphi(B)} u_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} \quad (8.3)$$

where

$$\mu = \bar{\mu} / \varphi(B) = \bar{\mu} / (1 - \sum_{j=1}^p \varphi_j), \quad (8.4)$$

$$\frac{\theta(B)}{\varphi(B)} := \psi(B) = \sum_{j=0}^{\infty} \psi_j B^j. \quad (8.5)$$

The coefficients  $\psi_j$  are obtained by solving the equation

$$\varphi(B) \psi(B) = \theta(B). \quad (8.6)$$

In this case, we also have:

$$\mathbb{E}(X_{t-j} u_t) = 0, \quad \forall j \geq 1. \quad (8.7)$$

The  $\psi_j$  coefficients may be computed in the following way (setting  $\theta_0 = -1$ ):

$$\left( 1 - \sum_{k=1}^p \varphi_k B^k \right) \left( \sum_{j=0}^{\infty} \psi_j B^j \right) = 1 - \sum_{j=1}^q \theta_j B^j = - \sum_{j=1}^q \theta_j B^j \quad (8.8)$$

hence

$$\begin{aligned} \varphi(B)\psi_j &= -\theta_j \quad \text{for } j=0, 1, \dots, q \\ &= 0 \quad \text{for } j \geq q+1, \end{aligned} \tag{8.9}$$

where  $\psi_j = 0$ , for  $j < 0$ . Consequently,

$$\begin{aligned} \psi_j &= \sum_{k=1}^p \varphi_k \psi_{j-k} - \theta_j, \quad \text{for } j=0, 1, \dots, q \\ &= \sum_{k=1}^p \varphi_k \psi_{j-k}, \quad \text{for } j \geq q+1, \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} \psi_0 &= 1, \\ \psi_1 &= \varphi_1 \psi_0 - \theta_1 = \varphi_1 - \theta_1, \\ \psi_2 &= \varphi_1 \psi_1 + \varphi_2 \psi_0 - \theta_2 = \varphi_1 \psi_1 + \varphi_2 - \theta_2 = \varphi_1^2 - \varphi_1 \theta_1 + \varphi_2 - \theta_2, \\ &\vdots \\ \psi_j &= \sum_{k=1}^p \varphi_k \psi_{j-k}, \quad j \geq q+1. \end{aligned} \tag{8.11}$$

The  $\psi_j$  coefficients behave like the autocorrelations of an AR( $p$ ) process, except for the initial coefficients  $\psi_1, \dots, \psi_q$ .

## 8.2. Autocovariances and autocorrelations

Suppose:

a) the process  $X_t$  is second-order stationary with  $\sum_{j=1}^p \varphi_j \neq 1$  ; (8.12)

b)  $\mathbb{E}(X_{t-j}u_t) = 0, \forall j \geq 1$ .

By the stationarity assumption,

$$\mathbb{E}(X_t) = \mu, \forall t, \tag{8.13}$$

hence

$$\mu = \bar{\mu} + \sum_{j=1}^p \varphi_j \mu \tag{8.14}$$

and

$$\mathbb{E}(X_t) = \mu = \bar{\mu} / \left( 1 - \sum_{j=1}^p \varphi_j \right). \tag{8.15}$$

The mean is the same as in the case of a pure AR( $p$ ) process. The MA( $q$ ) component of the model has no effect on the mean. Let us now rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^p \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^q \theta_j u_{t-j} \tag{8.16}$$

where  $\tilde{X}_t = X_t - \mu$ . Consequently,

$$\tilde{X}_{t+k} = \sum_{j=1}^p \varphi_j \tilde{X}_{t+k-j} + u_{t+k} - \sum_{j=1}^q \theta_j u_{t+k-j}, \quad (8.17)$$

$$\mathbb{E}(\tilde{X}_t \tilde{X}_{t+k}) = \sum_{j=1}^p \varphi_j \mathbb{E}(\tilde{X}_t \tilde{X}_{t+k-j}) + \mathbb{E}(\tilde{X}_t u_{t+k}) - \sum_{j=1}^q \theta_j \mathbb{E}(\tilde{X}_t u_{t+k-j}), \quad (8.18)$$

$$\gamma(k) = \sum_{j=1}^p \varphi_j \gamma(k-j) + \gamma_{xu}(k) - \sum_{j=1}^q \theta_j \gamma_{xu}(k-j), \quad (8.19)$$

where

$$\gamma_{xu}(k) = \mathbb{E}(\tilde{X}_t u_{t+k}) = 0, \quad \text{if } k \geq 1, \\ \neq 0, \quad \text{if } k \leq 0, \quad (8.20)$$

$$\gamma_{xu}(0) = \mathbb{E}(\tilde{X}_t u_t) = \sigma^2.$$

For  $k \geq q+1$ ,

$$\gamma(k) = \sum_{j=1}^p \varphi_j \gamma(k-j), \quad (8.21)$$

$$\rho(k) = \sum_{j=1}^p \varphi_j \rho(k-j). \quad (8.22)$$

The variance is given by

$$\gamma(0) = \sum_{j=1}^p \varphi_j \gamma(j) + \sigma^2 - \sum_{j=1}^q \theta_j \gamma_{xu}(-j), \quad (8.23)$$

hence

$$\gamma(0) = \left[ \sigma^2 - \sum_{j=1}^q \theta_j \gamma_{xu}(-j) \right] / \left[ 1 - \sum_{j=1}^p \varphi_j \rho(j) \right]. \quad (8.24)$$

In operational notation, the autocovariances satisfy the equation

$$\varphi(\mathbf{B})\gamma(k) = \theta(\mathbf{B})\gamma_{xu}(k), \quad k \geq 0, \quad (8.25)$$

where  $\gamma(-k) = \gamma(k)$ ,  $\mathbf{B}^j \gamma(k) := \gamma(k-j)$  and  $\mathbf{B}^j \gamma_{xu}(k) := \gamma_{xu}(k-j)$ . In particular,

$$\varphi(\mathbf{B})\gamma(k) = 0, \quad \text{for } k \geq q+1, \quad (8.26)$$

$$\varphi(\mathbf{B})\rho(k) = 0, \quad \text{for } k \geq q+1. \quad (8.27)$$

To compute the autocovariances, we can solve the equations (8.19) for  $k=0, 1, \dots, p$ , and then apply (8.21). The autocorrelations of an process ARMA( $p, q$ ) process behave like those of an AR( $p$ ) process, except that initial values are modified.

**Example 8.1** Consider the ARMA(1, 1) model:

$$X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t - \theta_1 u_{t-1}, \quad |\varphi_1| < 1 \quad (8.28)$$

$$\tilde{X}_t - \varphi_1 \tilde{X}_{t-1} = u_t - \theta_1 u_{t-1} \quad (8.29)$$

where  $\tilde{X}_t = X_t - \mu$ . We have

$$\gamma(0) = \varphi_1 \gamma(1) + \gamma_{xu}(0) - \theta_1 \gamma_{xu}(-1), \quad (8.30)$$

$$\gamma(1) = \varphi_1 \gamma(0) + \gamma_{xu}(1) - \theta_1 \gamma_{xu}(0) \quad (8.31)$$

and

$$\gamma_{xu}(1) = 0, \quad (8.32)$$

$$\gamma_{xu}(0) = \sigma^2, \quad (8.33)$$

$$\begin{aligned} \gamma_{xu}(-1) &= \mathbb{E}(\tilde{X}_t u_{t-1}) = \varphi_1 \mathbb{E}(\tilde{X}_{t-1} u_{t-1}) + \mathbb{E}(u_t u_{t-1}) - \theta_1 \mathbb{E}(u_{t-1}^2) \\ &= \varphi_1 \gamma_{xu}(0) - \theta_1 \sigma^2 = (\varphi_1 - \theta_1) \sigma^2 \end{aligned} \quad (8.34)$$

Thus,

$$\begin{aligned} \gamma(0) &= \varphi_1 \gamma(1) + \sigma^2 - \theta_1 (\varphi_1 - \theta_1) \sigma^2 \\ &= \varphi_1 \gamma(1) + [1 - \theta_1 (\varphi_1 - \theta_1)] \sigma^2, \end{aligned} \quad (8.35)$$

$$\begin{aligned} \gamma(1) &= \varphi_1 \gamma(0) - \theta_1 \sigma^2 \\ &= \varphi_1 \{ \varphi_1 \gamma(1) + [1 - \theta_1 (\varphi_1 - \theta_1)] \sigma^2 \} - \theta_1 \sigma^2, \end{aligned} \quad (8.36)$$



hence

$$\begin{aligned}
\gamma(1) &= \{\varphi_1[1 - \theta_1(\varphi_1 - \theta_1)] - \theta_1\}\sigma^2 / (1 - \varphi_1^2) \\
&= \{\varphi_1 - \theta_1\varphi_1^2 + \varphi_1\theta_1^2 - \theta_1\}\sigma^2 / (1 - \varphi_1^2) \\
&= (1 - \theta_1\varphi_1)(\varphi_1 - \theta_1)\sigma^2 / (1 - \varphi_1^2).
\end{aligned} \tag{8.37}$$

Similarly,

$$\begin{aligned}
\gamma(0) &= \varphi_1\gamma(1) + [1 - \theta_1(\varphi_1 - \theta_1)]\sigma^2 \\
&= \varphi_1 \frac{(1 - \theta_1\varphi_1)(\varphi_1 - \theta_1)\sigma^2}{1 - \varphi_1^2} + [1 - \theta_1(\varphi_1 - \theta_1)]\sigma^2 \\
&= \frac{\sigma^2}{1 - \varphi_1^2} \{ \varphi_1(1 - \theta_1\varphi_1)(\varphi_1 - \theta_1) + (1 - \varphi_1^2)[1 - \theta_1(\varphi_1 - \theta_1)] \} \\
&= \frac{\sigma^2}{1 - \varphi_1^2} \{ \varphi_1^2 - \theta_1\varphi_1^3 + \varphi_1^2\theta_1^2 - \varphi_1\theta_1 + 1 - \varphi_1^2 - \theta_1\varphi_1 + \theta_1\varphi_1^3 + \theta_1^2 - \varphi_1^2\theta_1^2 \} \\
&= \frac{\sigma^2}{1 - \varphi_1^2} \{ 1 - 2\varphi_1\theta_1 + \theta_1^2 \}.
\end{aligned} \tag{8.38}$$

Thus,

$$\gamma(0) = (1 - 2\varphi_1\theta_1 + \theta_1^2)\sigma^2 / (1 - \varphi_1^2), \tag{8.39}$$

$$\gamma(1) = (1 - \theta_1\varphi_1)(\varphi_1 - \theta_1)\sigma^2 / (1 - \varphi_1^2), \tag{8.40}$$

$$\gamma(k) = \varphi_1\gamma(k-1), \text{ for } k \geq 2. \tag{8.41}$$

## 9. Invertibility

A second-order stationary  $AR(p)$  process in  $MA(\infty)$  form. Similarly, any second-order stationary  $ARMA(p, q)$  process can also be expressed as  $MA(\infty)$  process. By analogy, it is natural to ask the question: can an  $MA(q)$  or  $ARMA(p, q)$  process be represented in a autoregressive form?

Consider the  $MA(1)$  process

$$X_t = u_t - \theta_1 u_{t-1}, t \in \mathbb{Z}, \quad (9.1)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$  and  $\sigma^2 > 0$ . We see easily that

$$\begin{aligned} u_t &= X_t + \theta_1 u_{t-1} \\ &= X_t + \theta_1 (X_{t-1} + \theta_1 u_{t-2}) \\ &= X_t + \theta_1 X_{t-1} + \theta_1^2 u_{t-2} \\ &= \sum_{j=0}^n \theta_1^j X_{t-j} + \theta_1^{n+1} u_{t-n-1} \end{aligned} \quad (9.2)$$

and

$$\mathbb{E} \left[ \left( \sum_{j=0}^n \theta_1^j X_{t-j} - u_t \right)^2 \right] = \mathbb{E} \left[ \left( \theta_1^{n+1} u_{t-n-1} \right)^2 \right] = \theta_1^{2(n+1)} \sigma^2 \xrightarrow{n \rightarrow \infty} 0 \quad (9.3)$$

provided  $|\theta_1| < 1$ . Consequently, the series  $\sum_{j=0}^n \theta_1^j X_{t-j}$  converges in *q.m.* to  $u_t$  if  $|\theta_1| < 1$ . In other words,

when  $|\theta_1| < 1$ , we can write

$$\sum_{j=0}^{\infty} \theta_1^j X_{t-j} = u_t, t \in \mathbb{Z}, \quad (9.4)$$

or

$$(1 - \theta_1 B)^{-1} X_t = u_t, t \in \mathbb{Z} \quad (9.5)$$

where  $(1 - \theta_1 B)^{-1} = \sum_{j=0}^{\infty} \theta_1^j B^j$ . The condition  $|\theta_1| < 1$  is equivalent to having the roots of the equation  $1 - \theta_1 z = 0$  outside the unit circle. If  $\theta_1 = 1$ ,

$$X_t = u_t - u_{t-1} \quad (9.6)$$

and the series

$$(1 - \theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} \theta_1^j X_{t-j} = \sum_{j=0}^{\infty} X_{t-j} \quad (9.7)$$

does not converge, for  $\mathbb{E}(X_{t-j}^2)$  does not converge to 0 as  $j \rightarrow \infty$ . Similarly, if  $\theta_1 = -1$ ,

$$X_t = u_t + u_{t-1} \quad (9.8)$$

and the series

$$(1 - \theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} (-1)^j X_{t-j} \quad (9.9)$$

does not converge either. These models are not invertible.

**Theorem 9.1** INVERTIBILITY CONDITION FOR A MA PROCESS. Let  $\{X_t : t \in \mathbb{Z}\}$  be a second-order stationary process such that

$$X_t = \mu + \theta(B)u_t \quad (9.10)$$

where  $\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$ . Then the process  $X_t$  satisfies an equation of the form

$$\sum_{j=0}^{\infty} \bar{\phi}_j X_{t-j} = \bar{\mu} + u_t \quad (9.11)$$

iff the roots of the polynomial  $\theta(z)$  are outside the unit circle. Further, when the representation (9.11) exists, we have:

$$\bar{\phi}(B) = \theta(B)^{-1}, \bar{\mu} = \theta(B)^{-1} \mu = \mu / \left(1 - \sum_{j=1}^q \theta_j\right). \quad (9.12)$$

**Corollary 9.2** INVERTIBILITY CONDITION FOR AN ARMA PROCESS. Let  $\{X_t : t \in \mathbb{Z}\}$  be a second-order stationary ARMA process that satisfies the equation

$$\varphi(B)X_t = \bar{\mu} + \theta(B)u_t \quad (9.13)$$

where  $\varphi(B) = 1 - \varphi_1 B - \dots - \varphi_p B^p$  and  $\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$ . Then the process  $X_t$  satisfies an equation of the form

$$\sum_{j=0}^{\infty} \bar{\phi}_j X_{t-j} = \bar{\bar{\mu}} + u_t \quad (9.14)$$

iff the roots of polynomial  $\theta(z)$  are outside the unit circle. Further, when the representation (9.14) exists, we have:

$$\bar{\phi}(B) = \theta(B)^{-1} \varphi(B), \quad \bar{\bar{\mu}} = \theta(B)^{-1} \bar{\mu} = \mu / \left( 1 - \sum_{j=1}^q \theta_j \right). \quad (9.15)$$

## 10. Wold representation

We have seen that all second-order ARMA processes can be written in a causal MA( $\infty$ ) form. This property indeed holds for all second-order stationary processes.

**Theorem 10.1** WOLD REPRESENTATION OF WEAKLY STATIONARY PROCESSES. *Let  $\{X_t, t \in \mathbb{Z}\}$  be a second-order stationary process such that  $\mathbb{E}(X_t) = \mu$ . Then  $X_t$  can be written in the form*

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} + v_t \quad (10.1)$$

where  $\{u_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \sigma^2)$ ,  $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ ,  $\mathbb{E}(u_t X_{t-j}) = 0$ ,  $\forall j \geq 1$ , and  $\{v_t : t \in \mathbb{Z}\}$  is a deterministic process such that  $\mathbb{E}(v_t) = 0$  and  $\mathbb{E}(u_s v_t) = 0$ ,  $\forall s, t$ . Further, if  $\sigma^2 > 0$ , the sequences  $\{\psi_j\}$  and  $\{u_t\}$  are unique, and

$$u_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t-1}, \tilde{X}_{t-2}, \dots) \quad (10.2)$$

where  $\tilde{X}_t = X_t - \mu$ .

PROOF. See Anderson (1971, Section 7.6.3, pp. 420-421) and Hannan (1970, Chapter III, Section 2, Theorem 2, pp. 136-137).  $\square$

If  $\mathbb{E}(u_t^2) > 0$  in Wold representation, we say the process  $X_t$  is *regular*.  $v_t$  is called the deterministic component of the process while  $\sum_{j=0}^{\infty} \psi_j u_{t-j}$  is its *indeterministic component*. When  $v_t = 0, \forall t$ , the process  $X_t$  is said to be *strictly indeterministic*.

**Corollary 10.2** FORWARD WOLD REPRESENTATION OF WEAKLY STATIONARY PROCESSES. Let  $\{X_t : t \in \mathbb{Z}\}$  be second-order a stationary process such that  $\mathbb{E}(X_t) = \mu$ . Then  $X_t$  can be written in the form

$$X_t = \mu + \sum_{j=0}^{\infty} \tilde{\psi}_j \bar{u}_{t+j} + \bar{v}_t \quad (10.3)$$

where  $\{\bar{u}_t : t \in \mathbb{Z}\} \sim \text{WN}(0, \bar{\sigma}^2)$ ,  $\sum_{j=0}^{\infty} \tilde{\psi}_j^2 < \infty$ ,  $\mathbb{E}(\bar{u}_t X_{t+j}) = 0, \forall j \geq 1$ , and  $\{\bar{v}_t : t \in \mathbb{Z}\}$  is a deterministic (with respect to  $\bar{v}_{t+1}, \bar{v}_{t+2}, \dots$ ) such that  $\mathbb{E}(\bar{v}_t) = 0$  and  $\mathbb{E}(\bar{u}_s \bar{v}_t) = 0, \forall s, t$ . Further, if  $\bar{\sigma}^2 > 0$ , the sequences  $\{\tilde{\psi}_j\}$  and  $\{\bar{u}_t\}$  are uniquely defined, and

$$\bar{u}_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t+1}, \tilde{X}_{t+2}, \dots) \quad (10.4)$$

where  $\tilde{X}_t = X_t - \mu$ .

PROOF. The result follows on applying Wold theorem to the process  $Y_t := X_{-t}$  which is also second-order stationary.  $\square$

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